On Schwarz genus, Lusternik–Schnirelmann category, and topological complexity*

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1 Introduction

In 2003 Michael Farber introduced a new numerical invariant - topological complexity of a space. The goal of this research was a “topologization” of certain situation in robotics, mainly, in motion problem for robots. Currently, we have a developed branched theory with numerous versions of topological complexity, as well as several interesting applications.

Here I write a survey concerning these ideas. Since topological complexity is a special case of Schwarz genus, and it is a close relative to Lusternik–Schniremann category, we expose all these three concepts together.

2 Preliminaries

The word “space” always means “completely normal topological space” unless something other is said explicitly.
The word “smooth” always means C^∞ (function or manifold).
All maps are assumed to be continuous unless something other is said explicitly. The identity map $X \to X$ is denoted by id_X.
All functional spaces of the form Y^X are assumed to be equipped with compact-open topology.
We use notation $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of integer, real, and complex numbers, respectively,
We denote by I the closed interval $[0, 1]$.
We use the notation $:=$ as “equal by definition”.
We use the sign $\simeq$ for homotopy of maps or homotopy equivalences of spaces
We use the abbreviation “iff” for “if and only if”;
“Fibration” means a Hurewicz fibration over a path connected finite CW base, unless some other is said explicitly.

3 Schwarz genus, or sectional category

Recall that a *section* of a map $f : X \to Y$ is a map $s : Y \to X$ such that $fs = \mathrm{id}_Y$.

Definition 3.1. Let $\xi = \{p : E \to B\}$ be a fibration over a base B. A *Schwarz genus*, or *sectional category* $\operatorname{secat} \xi$ of ξ is a minimal number k such that there exists an open covering $\{U_0, \dots, U_k\}$ of B with the following property: for each $i = 0, \dots, k$ the fibration

$$p^{-1}(U_i) \to U_i$$

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has a section. (In other words, s_i is a local section of p.) We also put $\operatorname{secat}\xi = -1$ if $B = \varnothing$.

This concept was introduced by Albert Schwarz in [S] under the name "genus". James [J1] proposes to replace the overworked term "genus" by "sectional category". (Of course, the word "category" is also overworked, but the adjective "sectional" softens the situation.)

Remark 3.2. The original definition by Schwarz differs by 1 from 3.1. However, now the Definition 3.1 is commonly accepted, see e.g. [CLOT].

Recall that any two fibers of a fibration $\xi : E \to B$ are homotopy equivalent (since B is path connected, see preliminaries), and we define the homotopy class of the fibers by the *homotopy fiber of* ξ. Sometimes we write $\xi = \{F \to E \to B\}$ where F is the homotopy fiber of ξ.

Given a map $f : Y \to X$, define the *Serre fibrational substitute* of f as follows. Put

$$E = \{(\omega, x) | \omega \in Y^I, x \in X, \omega(1) = f(x)\}$$

and define $\widehat{f} : E \to X$ as $\widehat{f}(\omega, x) = x$ and $\varphi : E \to Y, \varphi(\omega, x) = \omega(1)$. It is easy to see that $\widehat{f} : E \to X$ is a fibration, $\widehat{f} = f\varphi$, and φ is a homotopy equivalence, see [Sp].

We start with elementary facts. Given two fibrations

$$\xi = \{p : E \to B\} \text{ and } \xi' = \{p' : E' \to B'\},$$

consider their product $\xi \times \xi' = \{p \times p' : E \times E' \to B \times B'\}$.

Proposition 3.3 ([S, Prop. 21])**.** $\operatorname{secat}(\xi \times \xi') \le \operatorname{secat}\xi + \operatorname{secat}\xi'$.

This theorem dates back to Bassi [Bas], who proved the similar inequality for Lusternik-Schnirelmann category.
Given a fibration $\xi = \{p : E \to B\}$ and a map $f : X \to B$, consider the induced fibration $f^*\xi$ over X.

Proposition 3.4 ([S, Prop. 7])**.** $\operatorname{secat} f^*\xi \le \operatorname{secat}\xi$.

Theorem 3.5 ([S, Theorem 4])**.** Given a commutative ring R and fibration $\xi = \{p : E \to B\}$, suppose that there are cohomology classes $u_i \in H^*(B; R), i = 1, \dots, k$ such that $p^*u_i = 0, i = 1, \dots, k$ and that $u_1 \smile \cdots \smile u_k \ne 0$. Then $\operatorname{secat}\xi \ge k$.

Theorem 3.5 hints (motivates) the following definition.

Definition 3.6. Given a fibration $\xi = \{p : E \to B\}$ and a commutative ring R, define the *cup-length* of ξ, denoted by $\operatorname{cl}_R(\xi)$ or $\operatorname{cl}_R(p)$, to be the maximal number m such that there exist cohomology classes $u_i \in H^*(B; R), i = 1, \dots, m$ such that $p^*u_i = 0, i = 1, \dots, m$ and that $u_1 \smile \cdots \smile u_m \ne 0$.

So, Theorem 3.5 claims that $\operatorname{secat}\xi \ge \operatorname{cl}(\xi)$.

Remark 3.7. In fact, 3.5 states a more general claim: we can consider the local coefficient systems $A_i, i = 1, \dots, k$ on B and cohomology classes $u_i \in H^*(X; A_i)$, and get the analog (and generalization) of the above mention claim. In this case we have

$$u_1 \smile \cdots \smile u_k \in H^*(B; A_1 \otimes \cdots \otimes A_k).$$

To go further, we need to recall what is the iterated fiberwise join, see [Hu]. The join $X * Y$ of two spaces X and Y is a quotient space

$$(X \times Y \times I) \sim$$

where the equivalence relation $\sim$ is generated by the equivalences
$(x, 0, y_1) \sim (x, 0, y_2)$ for all $x \in X, y_1, y_2 \in Y$ and $(x_1; 1; y) \sim (x_2; 1; y)$ for all $y \in Y, x_1, x_2 \in X$.
More generally, given two maps $f : X \to Z$ and $g : Y \to Z$, we can construct the join $f *_Z g : X *_Z Y \to Z$ of f and g over Z by setting

$$X *_Z Y = \{[x, t, y] \in X * Y | f(x) = g(y)\} \text{ and } (f *_Z g)([x, t, y]) = f(x).$$

We can iterate the join construction. In particular, given a fibration $\xi = \{p : E \to X\}$ with the homotopy fiber F, we have the n-fold fiberwise join $\xi^{(n)} = E *_X * \cdots *_X E$ over X. It is a fibration whose homotopy fiber is F^{*n}, the n-fold join of F with itself.
There is a more explicit construction of $\xi^{(n)}$ as $\xi^{(n)} = \{(t_1e_1 + \cdots t_ne_n)\}$ where $e_1, \dots e_n$ are in same fiber of ξ and $t_1, \dots, t_n$ are non-negative real numbers such that $t_1 + \cdots + t_n = 1$. The identifications are such that t_ie_i is independent of e_i when $t_i = 0$.

Theorem 3.8 ([S, Theorem 3])**.** For any fibration ξ, $\operatorname{secat}\xi < n$ iff the fiberwise join $\xi^{(n)}$ has a section.

Theorem 3.9 ([S, Theorem 5])**.** Given a fibration $\xi = \{p : E \to B\}$ with the homotopy fiber F, assume that F is $(k-1)$ connected. Then

$$\operatorname{secat}\xi < \frac{\dim B + 1}{k+1} .$$

Below we consider three important examples of sectional category: Lusternik–Schnirelmann category, topological complexity, and higher (or sequential) topological complexity. It is worth saying that LS category appeared in 1929, see [LS1], the Schwarz's paper [S] appeared in 1958 (in Russian), and topological compexity appeared in 2003, [F1].

4 Lusternik–Schnirelmann category

A good source for the Lusternik–Schnirelmann category is [CLOT].

Definition 4.1. Given a map $f : X \to Y$, an *f-categorical set* is an open subset U of X such that $f_{|U} : U \to Y$ is null-homotopic. An *f-categorical covering* is a covering $\{U_i\}$ of X such that every set U_i is f-categorical. The Lusternik–Schnirelmann category $\operatorname{cat} f$ of f is defined to be the minimal number k such that there exists f-categorical covering $\{U_0, U_1, \dots, U_k\}$ of X. If no such k exists, we write $\operatorname{cat} f = \infty$. Furthermore, we set $\operatorname{cat} X := \operatorname{cat}(\mathrm{id}_X)$.

Proposition 4.2 ([BG])**.** For every diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we have: $\operatorname{cat}(gf) \le \min\{\operatorname{cat} f, \operatorname{cat} g\}$.

Proposition 4.3 ([CLOT, Theorem 1:30])**.** If $X \simeq Y$ then $\operatorname{cat} X = \operatorname{cat} Y$. In other words, $\operatorname{cat} X$ is a homotopy invariant of X.

Proposition 4.4. For every fibration $\xi = \{p : E \to B\}$ we have $\operatorname{secat} \xi \leq \operatorname{cat} B$.

For the proof, note that, for any subset A of B that is contractible in B, the fibration ξ admits a section over A.

The main application of the Lusternik–Schnirelmann category is the following.

Theorem 4.5 ([LS1, LS2])**.** Let $f : M \to \mathbb{R}$ be a smooth function on a closed smooth manifold M, and $\operatorname{Crit} f$ denote the number of critical points of f. Then $\operatorname{cat} M + 1 \leq \operatorname{Crit} f$.

Remark 4.6. Historically, Lusternik and Schnirelmann attacked the Poincarè conjecture (1905) that every Riemannian manifold with the topology of a 2-dimensional sphere has at least three closed geodesics that form simple closed curves without self-intersections. (See [B] for detailed publication.) By the way, a general ellipsoid gives us exactly 3 geodesics. Furthermore, we can regard geodesics as critical points of the length function on a suitable spaces of curved. So, it makes sense to study critical points on a manifold as a preliminary research. This is how Lusternik and Schnirelmann Theorem 4.5 appeared.

The following theorem relates Lusternik–Schnirelmann category to sectional category. Given a path connected space X and $x_0 \in X$, put $PX = \{\omega \in X^I, \omega(0) = x_0\}$ and define the fibration

$$\{\eta = \eta_X : PX \to X, p(\omega) = \omega(1)\}.$$

It worth noting that the homotopy fiber of η_X is ΩX.

Theorem 4.7 ([S, Theorem 18])**.** We have $\operatorname{cat} X = \operatorname{secat} \eta$.

Because of this and Proposition 3.3, we get the following claim.

Corollary 4.8. We have $\operatorname{cat}(X \times Y) \leq \operatorname{cat} X + \operatorname{cat} Y$ if the spaces X and Y are path connected.

Let $\eta_X^{*k} : P_k X \to X$ denote the k-fold fiberwise join $\eta_X * \cdots * \eta_X$ over X. Theorem 3.8 implies the following claim.

Corollary 4.9. The fibration η_X^{*k} has a section iff $\operatorname{cat} X < k$.

There is another (but homotopy equivalent) description of the fibration $\eta_X^{*k} : P_k X \to X$ given by Ganea, see [G], the so-called fiber-cofiber construction. See [CLOT, Definition 1.59 and Example 1.61].

Corollary 4.10. If X is $(k-1)$-connected with $k > 0$, then

$$\operatorname{cat} X < \frac{\dim X + 1}{k}. \text{ In particular, } \operatorname{cat} X \leq \frac{\dim X}{k}.$$

This follows from Theorem 3.9 if we recall that the homotopy fiber ΩX of η_X is $(k-2)$-connected.

5 Cup-length

Definition 5.1. Given a commutative ring R and a space X, define the *cup-length* of X with coefficients in R, denoted by $\mathrm{cl}_R(X)$ as the maximal number k such that $u_1 \smile \cdots \smile u_k \neq 0$ for $u_i \in \widetilde{H}^*(X; R)$.

In other words, for X path connected we have $\mathrm{cl}(X) := \mathrm{cl}(\eta_X)$ in accordance with Definition 3.6.

Theorem 3.5 implies the following corollary.

Corollary 5.2 ([FE])**.** We have $\mathrm{cl}_R(X) \leq \mathrm{cat}\, X$.

Pay attention that the paper [FE] appears before the paper [S]. Moreover, [FE] was stated in terms of homology and intersection of cycles: at that time the cohomology language was not "up in the air".

Remark 5.3. More generally, the corollary Corollary 5.2 holds if we use local coefficients systems as in Remark 3.7. Moreover, we can consider classes $u_i \in E^*(X)$ for a multiplicative cohomology theory (spectrum) E and state an obvious analog of Corollary 5.2.

We explain the proof of Corollary 5.2 for X path connected. Apply Theorem 3.5 to the case when ξ is the fibration $\eta = \{p : PX \to X\}$. Now note that $p^* u_i = 0$ because PX is contractible,

Remark 5.4. It is instructive and nice to present the following sketch of the proof of Corollary 5.2. Let $\mathrm{cat}\, X = k$. We prove that $\mathrm{cl}(X) \leq k$ by proving that $u_0 \smile u_1 \smile \cdots \smile u_k = 0$ for all $u_0, \ldots, u_k$. Indeed, let $\{U_0, \ldots, U_k\}$ be a categorical covering for X. We have:

$$u_{0|U_0} = 0,\ (u_0 \smile u_1)_{|(U_0 \cup U_1)} = 0, \ldots,$$
$$(u_0 \smile \cdots \smile u_k)_{|(U_0 \cup \cdots \cup U_k)} = 0.$$

But $U_0 \cup \cdots \cup U_k = X$.

Examples 5.5.

1. Real projective space $\mathbb{RP}^n$.
We know that $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ is the truncated polynomial ring

$$\mathbb{Z}/2[u]/(u^{n+1}), u \in H^1(\mathbb{RP}^n; \mathbb{Z}/2).$$

So, $\mathrm{cl}_{\mathbb{Z}/2}(\mathbb{RP}^n) \geq n$, and so $\mathrm{cat}(\mathbb{RP}^n) \geq n$ Thus, $\mathrm{cat}(\mathbb{RP}^n) = n$ because of Corollary 4.10.

2. Complex projective space $\mathbb{CP}^n$. We know that $H^*(\mathbb{C}P^n)$ is the truncated polynomial ring

$$\mathbb{Z}[x]/(x^{n+1}), x \in H^2(\mathbb{CP}^n).$$

So, $\mathrm{cl}_{\mathbb{Z}}(\mathbb{CP}^n) \geq n$, and so $\mathrm{cat}(\mathbb{CP}^n) \geq n$. Since $\mathbb{C}P_n$ is simply-connected, we conclude that $\mathrm{cat}(\mathbb{CP}^n) \leq n$ because of Corollary 4.10. Thus, $\mathrm{cat}(\mathbb{CP}^n) = n$.

3. Torus $\mathbb{T}^n$. We know that

$$H^*(\mathbb{T}^n) = \mathbb{Z}[x_1, \ldots, x_n|, \dim x_i = 1, x_i^2 = 0, i = 1, \ldots, n].$$

So, $\mathrm{cl}_{\mathbb{Z}}(\mathbb{T}^n) \geq n$, and so $\mathrm{cat}(\mathbb{T}^n) \geq n$. Furthermore, $\dim T^n = n$, and so $\mathrm{cat}(\mathbb{T}^n) \leq n$ because of Corollary 4.10. Thus, $\mathrm{cat}(\mathbb{T}^n) = n$.

4. More generally, let M^{2m} be a closed simply-connected symplectic manifold with a symplectic form ω. This means the ω is a closed non-degenerate 2-form on M. In particular, for the de Rham cohomology class $[\omega]$ we have

$$0 \neq [\omega]^m \in H^{2m}(M; \mathbb{R}).$$

So, $\mathrm{cl}_{\mathbb{R}}(M) \geq m$, and hence $\mathrm{cat}\, M \geq n$. Since M is simply-connected, we conclude that $\mathrm{cat}(M) \leq m$ by Corollary 4.10. Thus, $\mathrm{cat}\, M = m$.

5. Sphere $\mathbb{S}^n$. For completeness, we prove that $\mathrm{cat}(\mathbb{S}^n) = 1$. We have $\mathrm{cat}\, \mathbb{S}^n > 0$ because the sphere is not contractible. Furthermore, $\mathrm{cat}\, \mathbb{S}^n \leq 1$ because $\mathbb{S}^n$ is the union of two contractible subspaces (hemispheres). Thus, $\mathrm{cat}\, \mathbb{S}^n = 1$.

6 Category weight

Definition 6.1. Given a spectrum E and $u \in E^*(X)$, define a *category weight* of u, denoted by $\mathrm{wgt}\, u$, as

$$\mathrm{wgt}(u) = \sup\{k | \varphi(u^*) = 0 \text{ for every maps } \varphi : A \to X \text{ with } \mathrm{cat}\, \varphi < k\}.$$

The main application of category weight is the following generalization of Theorem 3.5: Given a commutative ring R and $u_i \in H^i(X; R)$, assume that $u_1 \smile \cdots \smile u_k \neq 0$. Then

$$\mathrm{cat}\, X \geq \sum_{i=1}^{k} \mathrm{wgt}\, u_i.$$

For the proof, see e.g. [R2, Theorem 1.12].

Remark 6.2. The idea ot category weight does back to Fadell and Husseini [FH]. They considered the definition as in 6.1, but used the *inclusions* $A \to X$. They used the term “category weight” and notation cwgt. However, their construction was not a homotopy invaiant, i.e. there examples of homotopy equivalences $f : X \to Y$ and an element $u \in H^*(Y)$ with $\mathrm{cwgt}(f^*(u)) \neq \mathrm{cwgt}\, u$. To see that Fadell–Husseini construction is not a homotopy invariant, see [R2, Corollary 1.9 and Example 1.10]. The homotopy invariant version was proposed by Rudyak [R2] (called “strict category weight”) and Strom [Str] (called “essential category weight”). Later both authors agreed to use the term ”category weight” and the notation wgt.

Examples 6.3. 1. ([FH]). Let p be an odd prime, let β be the Bockstein homomorphism $\beta : H^i(-; \mathbb{Z}/p) \to H^{i+1}(; \mathbb{Z}/p)$. Let

$$P^k : H^i(-; \mathbb{Z}/p) \to H^{i+2k(p-1)}(-; \mathbb{Z}/p)$$

be the Steenrod reduced power. Then we have $\mathrm{wgt}(\beta P^k(u)) \geq 2$ for $u \in H^{2k+1}(X; \mathbb{Z}/p))$ provided $u \neq 0$. In particular, $\mathrm{wgt}(\beta(u)) \geq 2$ for $u \in H^1(X; \mathbb{Z}/p)$, $u \neq 0$. (In fact, $\mathrm{wgt}(\beta(u)) = 2$.)
As an application, consider the lens space $L = S^{2n+1}/(\mathbb{Z}/p)$ where p is an odd prime. Let $u \in H^1(L; \mathbb{Z}/p)$ be a generator. Then

$$H^*(L; \mathbb{Z}/p) = \mathbb{Z}/p[u, \beta u]/(u^2, (\beta u)^n).$$

Hence, $u(\beta u)^{n-1} \neq 0$, and $\mathrm{wgt}(u(\beta u)^{n-1}) \geq 2n+1$ since $\mathrm{wgt}(\beta u) \geq 2$. So, $\operatorname{cat} L \geq 2n+1$. Thus $\operatorname{cat} L = 2n+1$ (Krasnosel'skii).

2. Let π be a discrete group and $B\pi = K(\pi, 1)$ be the classifying space for π. Then $\mathrm{wgt}\, u = k$ for every $u \in H^k(B\pi; G)$ and for every coefficient group G, see [R1, Str]. Since the infinite lens space $S^\infty/(\mathbb{Z}/p)$ is $K(\mathbb{Z}/p, 1)$, we have another proof of the Krasnosel'skii equality $\operatorname{cat} L = 2n+1$ from item **1**.

3. For every non-trivial (i.e. not containing zero) Massey product $\langle u, v, w\rangle$ and every $x \in H^*\langle u, v, w\rangle$ we have $\mathrm{wgt}\, x \geq 2$, [R2].

4. Let M^{2n} be a closed symplectic manifold with a symplectic form ω, and let $[\omega] \in H^2(M; \mathbb{R})$ be the de Rham cohomology class of ω. Suppose that $\displaystyle\int_{S^2} f^*\omega = 0$ for all smooth maps $f : S^2 \to M$ (the so-called sympletically aspherical manifolds). Then $\mathrm{wgt}[\omega] = 2$. This claim is an important ingredient for the proof of the Arnold conjecture, [RO].

Farber and Grant [FG2] generalized the notion of category weight (see Section 6) for sectional category as follows. Given a fibration $p : E \to B$ and cohomology class $u \in H^*(B)$, define the *category weight with respect to p*, denoted by wgt_p to be the maximal integer k such that $f^*u = 0$ for all maps $f : Y \to B$ with $\operatorname{secat} f^*(p) \leq k$. Here $f^*p : E' \to Y$ denotes the pull-back fibration of p along f.

Below in Section 7 we will consider *topological complexity* of a space S, denoted by $\mathrm{TC}(X)$ as the sectional category of a fibration

$$\pi : X^I \to X \times X, \pi(\alpha) = (\alpha(0), \alpha(1)).$$

In particular, we can introduce a notion of TC-weight as wgt_π. Among other applications of wgt_π, Farber and Grant got a lot of information of TC for some lens spaces. For more information on TC of lens spaces see [Gon, GZ].

7 Motion planning problem

As a good source on motion planning see [L, LV].
In robotics, motion planning problem (also known as the navigation problem or the piano mover's problem) is finding a path that moves the robot from the source to destination. One of mathematical descriptions of the problem looks as follows. Let X be a path connected topological space that we regard as the configuration space of a mechanical system. Points of X represent states of the system, and a continuous motion of the system can be regarded as a continuous path $\alpha : I \to X$. Here $\alpha(0)$ is the initial point and $\alpha(1)$ is the final point. Since X is path connected, we are able to move a point to any other one.
A *motion planning algorithm* on X is a rule that assigns a path $\alpha : I \to X$ to a pair $(\alpha(0), \alpha(1))$. More formally, consider the fibration

$$\zeta = \zeta_X = \{\pi : X^I \to X \times X, \pi(\alpha) = (\alpha(0), \alpha(1))\}.$$

In this way, a motion planning algorithm is a map (not necessarily continuous) $s : X \times X \to X^I$ such that $\pi s = \mathrm{id}_{X \times X}$. In other words, a motion planning algorithm is a section of ζ. It would be nice to work with *continuous* motion planning, i.e. to have the section s to be continuous. However, life is complicated: continuous sections exist if and only if the space X (as well as $X \times X$) is contractible, [F3]. So, it seems reasonable to consider a partition of $X \times X = \sqcup A_i$ so that every part A_i admits a continuous section s_i of ζ over A_i. This leads to the concept of *topological complexity*, which we turn to.

8 Topological complexity: a bridge from topology to robotics

Definition 8.1 (Farber[F3]). Let X be a path-connected CW space of finite type. *Topological complexity* of a space X (denoted by $\mathrm{TC}(X)$) is the sectional category of ζ_X. So, $\mathrm{TC}(X) = \operatorname{secat} \zeta_X$.

To relate topological complexity with motion planning problem, recall (in section 7) that we considered the partition of $X \times X = \sqcup A_i$ so that every part A_i admits a section of ζ over X_i. In particular, $A_i \cap A_j = \varnothing$ for $i \neq j$. The number of these parts shows how complicated can X be. How is this number related to $\mathrm{TC}(X)$? The answer is that, for X good enough, there is a partition $\{A_i\}$ as before whose number $\#A_i$ is equal to $\mathrm{TC}(X)+1$. To explain this in greater detail, we need to recall the notion of Euclidean Neighborhood Retract (ENR), see [D2]. For us, the advantage of ENR is the property that, given two open subsets A and B of an ENR, the difference $A \setminus B$ is also an ENR.

Theorem 8.2 ([F3]). Let X is a polyhedron in $\mathbb{R}^N$ with $\mathrm{TC}(X) = k$. There exist a motion planning algorithm $s : X \times X \to X^I$ and a partition $X \times X = A_0 \sqcup \cdots \sqcup A_k$ such that
(i) each A_i is an ENR;
(ii) for each i the restriction $s_{|A_i} : A_i \to X \times X$ is continuous.

Thus, if $\mathrm{TC}(X) = k$ then there exists a motion planning algorithm $s : X \times X \to X^I$ that has $k+1$ domains of continuity of s, and each domain of continuity is an ENR.

In Section 9 you will see many calculations of TC. To warm up the interest to the subject, we note that $\mathrm{TC}(S^{2n+1}) = 1$ and $\mathrm{TC}(S^{2n}) = 2$, see below. Pay attention to the remarkable contrast with the equality $\mathrm{cat}(S^n) = 1$ for all $n > 0$.

Another interesting point is the following theorem that relates TC with problem of immersion of projective space to $\mathbb{R}^m$.

Theorem 8.3 (Farber–Tabachnikov–Yuzvinsky[FTY]). For any $n \neq 1, 3, 7$ the number $\mathrm{TC}(\mathbb{RP})$ is equal to the smallest k such that $\mathbb{RP}^n$ admits an immersion to $\mathbb{R}^k$. Furthermore, for $n = 1, 3, 7$ we have $\mathrm{TC}(\mathbb{RP}^n)) = n$.

Remark 8.4. Here we present two notes on a difference between Lusternik–Schnirelman category and topological complexity.

1. Let $p : \widetilde{X} \to X$ be a covering map. Then $\mathrm{cat}\, \widetilde{X} \leq \mathrm{cat}\, X$. However, if $X = S^3 \times S^3 \vee S^1$ and $\widetilde{X}$ is the universal covering space of X then $\mathrm{TC}(X) \leq 3$ and $\mathrm{TC}(\widetilde{X}) \geq 4$), see [Dr1].

2. We have $\mathrm{cat}(X \vee Y) = \max\{\mathrm{cat}\, X, \mathrm{cat}\, Y\}$. However, $\mathrm{TC}(S^1) = 1$ while $\mathrm{TC}(S^1 \vee S^1) = 2$. More on $\mathrm{TC}(X \vee Y)$, see [Dr1].

It worth noting that many modifications of topological complexity have currently appeared. For instance, in [DD] the authors introduce geodesic complexity, by considering broken geodesics on a

Riemannian manifold. Another example: Mescher [Me] notes that a real robot has a shape, and hence one loses a lot of information by simply modeling the robot as a point in X. In this way, the author suggests to consider not points in PX but frames on a Riemannian manifold. One more example considers symmetric topological complexity [FG1]: the case when motion from one state A to another state B, prescribed by the algorithm, is the time reverse of the motion from B to A.

In this spirit, it is not unexpected that people suggest equivariant versions of topological complexity, by considering a group G acting on a space X. I do not discuss it here and quote the references [BK, CG, LM].

9 Higher, or sequential topological complexity

Rudyak [R4, BGRT] generalized the Farber's concept of topological complexity as follows. Given a space X, consider a fibration

$$\zeta_n = \zeta_{n,X} = \{e_n : X^I \to X^n\}$$
$$e_n(\alpha) = \left(\alpha(0), \alpha\left(\frac{1}{n-1}\right), \ldots, \alpha\left(\frac{n-2}{n-1}\right), \alpha(1)\right)$$

where $\alpha \in X^I$.

Definition 9.1 ([R4])**.** A *higher*, or *sequential topological complexity* of order n of a space X (denoted by $\mathrm{TC}_n(X)$) is the sectional category of ζ_n. So, $\mathrm{TC}_n(X) = \operatorname{secat} \zeta_n$.

It is easy to see that Farber's complexity $\mathrm{TC}(X)$ is equal to $\mathrm{TC}_2(X)$.

We show how TC_n is related to motion planning theory. Recall that $\mathrm{TC}(X)$ is related to motion planning algorithm when a robot moves from a point to another point. Similarly, $\mathrm{TC}_n(X)$ is related to motion planning problem whose input is not only an initial and final point but also $n-2$ intermediate additional points.

Now we establish some properties of TC_n. It worths to note that many properties of TC_n are obvious generalization of Farber's TC, and we exploit the ideas of Farber and his collaborators (see the references) in our research.

Proposition 9.2. $\mathrm{TC}_n(X \times Y) \le \mathrm{TC}_n(X) + \mathrm{TC}_n(Y)$.

It follows from Proposition 3.3.

Proposition 9.3. If X is $(k-1)$-connected then

$$\mathrm{TC}_n(X) \le \frac{n \dim X}{k}.$$

It follows from the definition of TC and Theorem 3.9, if note that the homotopy fiber of the diagonal $X \to X^n$ is $(k-2)$-connected.

Consider two fibrations $\xi = \{E \to B\}$ and $\xi' = \{E' \to B\}$ and a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Lemma 9.4. We have $\operatorname{secat}\xi \le \operatorname{secat}\xi'$. Moreover, if f is a fiber homotopy equivalence over B then $\operatorname{secat}\xi = \operatorname{secat}\xi'$.

Proof. If $s : A \to E$ is a local section of ξ over A then fs is a local section of ξ' over the same A. Hence, $\operatorname{secat}\xi \le \operatorname{secat}\xi'$. Furthermore, if f is a fiber homotopy equivalence over B then there exists a homotopy inverse $h : E' \to E$ over B, see [D1], and hence $\operatorname{secat}\xi \le \operatorname{secat}\xi'$. Thus, $\operatorname{secat}\xi = \operatorname{secat}\xi'$. Q.E.D.

Theorem 9.5. If the spaces X and Y are homotopy equivalent then $\mathrm{TC}_n(X) = \mathrm{TC}_n(Y)$. In other words, TC is a homotopy invariant.

For $n = 2$ see [F3, Theorem 3]. Now we show the proof for all n.

Proof. Take a homotopy equivalence $f : X \to Y$. Then f yields a morphism of fibrations

$$\zeta = \{X^I \to X^n\} \to \{Y^I \to Y^n\} = \zeta'.$$

The morphism $\zeta \to \zeta'$ can be decomposed as

$$\zeta \to f^*\zeta' \to \zeta'$$

where the morphism $\zeta \to f^*\zeta'$ induces the identity map id_{X^n} on bases. We have $\operatorname{secat}\zeta = \operatorname{secat} f^*\zeta'$ by Lemma 9.4, and $\operatorname{secat} g^*\zeta' \le \operatorname{secat}\zeta'$ by Proposition 3.4. So $\operatorname{secat}\zeta \le \operatorname{secat}\zeta'$. The existence of a homotopy inverse $Y \to X$ to f implies that $\operatorname{secat}\zeta = \operatorname{secat}\zeta'$. Q.E.D.

Theorem 9.6. For all n we have

$$\operatorname{cat} X^{n-1} \le \mathrm{TC}_n(X) \le \operatorname{cat} X^n \le \mathrm{TC}_{n+1}(X).$$

For the proof of first inequality see [BGRT, Prop.3.1]. The second inequality follows from Proposition 4.4.

In particular, $\mathrm{TC}_n(X) \le \mathrm{TC}_{n+1}(X)$.

Open Question 9.7. Do there exist a non-contractible space X and number n such that $\mathrm{TC}_n(X) = \mathrm{TC}_{n+1}(X)$?

Proposition 9.8. If X is not contractible then $\mathrm{TC}_n(X) \ge n - 1$ for all $n \ge 2$.

Indeed, $\mathrm{TC}_n(X) \ge \operatorname{cat} X^{n-1} \ge n - 1$. For the second inequality see [CLOT, Theorem 1.47].

Theorem 9.9. If G is a path-connected H-space (e.g. a topological group) then $\mathrm{TC}_n(G) = \operatorname{cat} G^{n-1}$.

For a topological group and $n = 2$ this is proved in [F2], for $n > 2$ see [BGRT]. For arbitrary H-spaces see [LupSch].

10 Some calculations and examples

Definition 10.1. Let X be a path connected CW space and $d = d_n : X \to X^n$ a diagonal map, $d(x) = (x, \cdots, x)$. A *zero-divior class* $x \in H^*(X^n)$ is the class with $d^*(x) = 0$. The *zero-divisor ideal* for X is the kernel of the map $d^* : H^*(X^n) \to H^*(X)$. A *zero-divisor cup-length* $\mathrm{zcl}(X) = \mathrm{zcl}_n(X)$ for X is a maximal number k such that

$$u_1 \smile \cdots \smile u_k \neq 0 \text{ and } d^*(u_i) = 0, \text{ for all } u_i \in H^*(X^n).$$

As usual, we can and shall use a generalization of zcl as in Remark 3.7.

Theorem 10.2. $\mathrm{TC}_n(X) \geq \mathrm{zcl}_n(X)$.

Proof. It follows from Theorem 3.5 if we replace fibration $p : E \to B$ in Theorem 3.5 by ζ_n and recall that the maps $e_n : X^I \to X^n$ and $d^n : X \to X^n$ are homotopy equivalent. Q.E.D.

Proposition 10.3. For any two path connected CW spaces X, Y of finite type with torsion free homology we have

$$\mathrm{zcl}_n(X \times Y) \geq \mathrm{zcl}_n(X) + \mathrm{zcl}_n(Y).$$

Proof. This follows from the definition of zcl and Künneth formula. Q.E.D.

The following theorem was proved in [F3] for $n = 2$ and in [R4] for $n > 2$.

Theorem 10.4. $\mathrm{TC}_n(S^{2k-1}) = n - 1$ and $\mathrm{TC}_n(S^{2k}) = n$ for all $n \geq 2$ and $k > 0$.

Proof. First, prove that $\mathrm{TC}_n(S^{2k+1}) = n - 1$. Take a unit tangent vector field $\mathbf{v}$ on S^{2k+1}. Given $x, y \in S^{2k+1}$, $y = -x$, let $[x, y]$ denote the path determined by the geodesic semicircle joining x to y and such that the vector $\mathbf{v}(x)$ is the direction of the semicircle at x. If $x \neq y$, let $[x, y]$ denote the path determined by the shortest geodesic from x to y.
Given $x_1, \ldots, x_n$ in S^{2k+1}, let $[x_1, x_2, \ldots, x_n]$ denote the contactenation of paths $[x_i, x_{i+1}]$, $i = 1, \ldots, n-1$, i.e. the path

$$[x_1, x_2, \ldots, x_n] := [x_1, x_2][x_2, x_3] \cdots\ [x_{n-1}, x_n] \text{ in } S^{2k+1}.$$

Now we have the (non-continuous) section

$$(S^{2k+1})^n \to (S^{2k+1})^I, \quad (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n].$$

Let $U_j \subset (S^{2k+1})^n, j = 0, \ldots, n-1$ consists of tuples $(x_1, \ldots, x_n)$ such that the family $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in S^{2k+1} has exactly j antipodal pairs (x_i, x_{i+1}). Now each U_j is a domain of continuity for the section $(S^{2k+1})^n \to (S^{2k+1})^I$. Hence, $\mathrm{TC}_n(S^{2k+1}) \leq n - 1$. On the other hand, $\mathrm{TC}_n(S^{2k+1}) \geq n - 1$ because of Proposition 9.8.

Now we prove that $\mathrm{TC}_n(S^{2k}) = n$. Take a generator $u \in H^{2k}(S^{2k}) = \mathbb{Z}$ and consider the element

$$w = \left(\sum_{i=1}^{n-1} 1 \otimes \cdots \otimes 1 \otimes u(i\text{th place}) \otimes 1 \otimes \cdots \otimes 1 \right) - 1 \otimes \cdots \otimes 1 \otimes (n-1)u$$

in $(H^{2k})^{\otimes n}((S^{2k})^n)$.

Note that w belongs to zero-divisor ideal. Furthermore,

$$w^{\smile n} = n!(1-n)u^{\otimes n} \neq 0.$$

Here we use the fact that $\dim S^{2k}$ is even. So, $\mathrm{zcl}_n(S^{2k}) \geq n$, and hence $\mathrm{TC}_n(S^{2k}) \geq n$ by Theorem 10.2. Finally, $\mathrm{TC}_n(S^{2k}) \leq n$ because the connectivity-dimension argument Proposition 9.3. Q.E.D.

Now we show a few examples.

Examples 10.5.

1. We claim that $\mathrm{zcl}_n(S^{2k-1}) = n-1$ and $\mathrm{zcl}_n(S^{2k}) = n$. In particular, $\mathrm{zcl}_n(S^{m)} = \mathrm{TC}_n(S^m)$ for all $m > 0$. For S^{2k} the clam is already proved in Theorem 10.4. Now we prove it for S^{2k-1}. Take a generator $v \in H^{2k-1}(S^{2k-1}) = \mathbb{Z}$. Let $p_i : (S^{2k-1})^n \to S^{2k-1}, i = 1, \ldots, n$ be the projection on i-th factor and put

$$v_i = p_i^*(v) \in H^{2k-1}((S^{2k-1})^n).$$

Let $d : S^{2k-1} \to (S^{2k-1})^n$ be the diagonal. Then $d^*(v_i) = v$, and $v_i - v_1$ is the zero divizor for all i. So

$$(v_2 - v_1) \smile \cdots \smile (v_n - v_1)$$

is the zero-divizor, and

$$(v_2 - v_1) \smile \cdots \smile (v_n - v_1) = v_2 \smile \cdots \smile v_n + v_1 V$$

for some $V \in H^{(2k-1)(n-1)}((S^{2k-1})^n)$. Hence

$$(v_2 - v_1) \smile \cdots \smile (v_n - v_1) \neq 0.$$

So, $\mathrm{zcl}_n(S^{2k-1}) \geq n-1$. Finally, $\mathrm{zcl}_n(S^{2k-1}) \leq \mathrm{TC}_n(S^{2k-1}) = n-1$, and thus $\mathrm{zcl}(S^{2k-1}) = n-1$.

2. More generally, for any path-connected CW space X and positive integers n and k we have $\mathrm{zcl}_n(X \times S^k) \geq \mathrm{zcl}_n(X) + n - 1$. This inequality can be improved to $\mathrm{zcl}_n(X \times S^k) \geq \mathrm{zcl}_n(X) + n$ provided k is even and $H^*(X)$ is torsion-free.
For the proof, see [BGRT, Theorem 3.10]

3. We claim that

$$\begin{aligned} \mathrm{TC}_n(S^{k_1} \times \cdots \times S^{k_m}) &= \mathrm{TC}_n(S^{k_1}) + \cdots + \mathrm{TC}_n(S^{k_m}) \\ &= m(n-1) + l \end{aligned}$$

where l is the number of even dimensional spheres.
The last equality follows from Theorem 10.4. Now, we have

$$\mathrm{TC}_n(S^{k_1} \times \cdots \times S^{k_m}) \leq \mathrm{TC}_n(S^{k_1}) + \cdots + \mathrm{TC}_n(S^{k_m})$$

by Proposition 9.2. On the other hand,

$$\begin{aligned} \mathrm{TC}_n(S^{k_1} \times \cdots \times S^{k_m}) &\geq \mathrm{zcl}_n(S^{k_1} \times \cdots \times S^{k_m}) \text{ (by Theorem 10.2)} \\ &\geq \mathrm{zcl}_n(S^{k_1}) + \cdots + \mathrm{zcl}_n(S^{k_m}) \text{ (by Proposition 10.3)} \\ &= \mathrm{TC}_n(S^{k_1}) + \cdots + \mathrm{TC}_n(S^{k_m}) \text{ (by item 1).} \end{aligned}$$

So we get the first equality.

4. Let X be a CW complex of finite type, and R a principal ideal domain. Take $u \in H^d(X;R)$ with $d > 0$, d even, and assume that the n-fold iterated self R-tensor power $u^m \otimes \cdots \otimes u^m \in (H^{md}(X;R))^{\otimes n}$ is an element of infinite additive order. Then $\mathrm{TC}_n(X) \geq mn$.
For the proof, see [BGRT, Theorem 3.14].

5. For every closed simply connected symplectic manifold M^{2m} we have $\mathrm{TC}_n(M) = mn$.
Indeed, $\mathrm{TC}_n(M) \geq mn$ because of item **4**, and $\operatorname{cat} M = m$ by Example 5.5, item 4. Thus

$$\mathrm{TC}_n(M) \leq n \operatorname{cat} M = mn.$$

6. $\mathrm{TC}_n(\mathbb{T}^k) = k(n-1)$.
This follows from Theorem 10.4 or Theorem 9.9.

We know that $\mathrm{TC}_n(X) \geq n-1$ for all X. Furthermore, if $\mathrm{TC}_2(X) = 1$ then X is homotopy equivalent to S^{2k-1}, [GLO]. However, we do not know if the similar fact holds for $n > 2$.

Open Question 10.6. Does the equality $\mathrm{TC}_n(X) = n-1, n > 2$ imply that X is homotopy equivalent to S^{2k-1}?

For an information on spaces of topological complexity 2, see [BR].

11 Monoidal topological complexity

Consider robot motion planning with the following property: if the initial position of a robot in the configuration space X coincides with the terminal position, then the algorithm keeps the robot still. This leads to the notion of monoidal topological complexity, [IS].

Definition 11.1. The monoidal topological complexity $\mathrm{TC}^M(X)$ is the least number m such that there exists a cover of $X \times X$ by $m+1$ open subsets $A_i, i = 0, \dots, m$ of $X \times X$ with the following property: each A_i has a local section $s_i : A_i \to PX$ for $\zeta = \{PX \to X \times X\}$ and, moreover: $s_i(x,x)$ is the constant path at x for all i and all $x \in X$.

Proposition 11.2 ([Dr1, IS])**.** We have

$$\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1$$

for all CW spaces X.

Open Question 11.3. Is it true that $\mathrm{TC}^M(X) = \mathrm{TC}(X)$ for all X?

12 Topological complexity of groups. Surfaces

Given a group π, let $K(\pi,1)$ be a path-connected space such that $\pi_1(K(\pi,1)) = \pi$ and $\pi_i(K(\pi,1)) = 0$ for $i > 1$ (so-called Eilenberg–MacLane space). It is well-known that the homotopy type of a CW space $K(\pi,1)$ is completely determined by π. Since TC is a homotopy invariant, we have a correctly defined algebraic invariant

$$\mathrm{TC}(\pi) := \mathrm{TC}(K(\pi,1)).$$

The simplest examples of $K(\pi,1)$-manifolds are the circle S^1, the orientable surfaces $S_g, g>0$, and non-orientable surfaces $N_g, g>1$. Here g denotes the genus of the surface.

We already know that $\mathrm{TC}(\mathbb{T}^2)=2$. Furthermore, $\mathrm{TC}(S_g)=4$ for $g>1$, [F2]. For the proof, look the chain of inequalities

$$4 \le \mathrm{zcl}(S_g) \le \mathrm{TC}(S_g) \le 2\dim S_g = 4.$$

(Pay attention that there is *non-normalized* complexity in [F2], so, the value at [F2] is one more than usual one.)

The case of non-orientable surfaces is much more complicated. We have $\mathrm{TC}(\mathbb{RP}^2)=3$, [FTY]. For $N_g, g>1$ we have $\mathrm{TC}(N_g)=4$, it is given by Cohen and Vandembroucq [CV] with an essential contribution of Costa and Faber [CF]. The case of $g=2$, the Klein bottle, was considered as a challenging problem, unlike the case $g>3$. The proof uses bar-construction (bar-resolution), crossed homomorphisms, and local coefficients.

Note that earlier Dranishnikov [DR2] proved that $\mathrm{TC}(N_g)=4$ for $g>3$ in a slightly different method.

Concerning TC_n of surfaces for $n>2$ we have the following result:

Theorem 12.1 ([GGGL])**.** Let S be a closed surface (orientable or not) different from the sphere and the torus. Then $\mathrm{TC}_n(S)=2n$ provided $n\ge 3$.

Surprisingly, the evaluation of $\mathrm{TC}_n(N_g)$ for $n>2$ is simpler than for $n=2$. Another surprise is that $\mathrm{TC}(\mathbb{RP}^2)=3$ while $\mathrm{TC}_n(\mathbb{RP}^2)=2n$ for $n>2$.

In [FGLO] authors deal with the problem of understanding $\mathrm{TC}(\pi)$ more deeply, by using Bredon equivariant cohomology. There are many interesting constructions that I do not want to present here, but I show the following concrete result. Let D be the class of all subgroups of the group $\pi\times\pi$ which are conjugate to the diagonal subgroup. Let $\mathrm{cd}_D(\pi\times\pi)$ denote the cohomological dimension of $\pi\times\pi$ with respect to the class D.

Theorem 12.2. $\mathrm{TC}(\pi)\le \max\{3, \mathrm{cd}_D(\pi\times\pi)\}$.

13 On sequences $\{\mathrm{TC}_n(X)\}_{n=2}^{\infty}$

This section is based on [R6, Section 15].

When we introduce the invariants TC_n, we have the following general question: Does the sequence $\{\mathrm{TC}_n\}$ gives more information than, say, the single invariant TC. The answer is positive. Indeed,

$$\mathrm{TC}(\mathbb{S}^2)=\mathrm{TC}(\mathbb{T}^2)=2,\ \mathrm{TC}_n(\mathbb{S}^2)=n,\ \mathrm{TC}_n(\mathbb{T}^2)=2n-2.$$

Another point of interest is the behavior of the sequence $\{\mathrm{TC}_n(X)\}$. Here we have the following proposition, [R6]

Proposition 13.1. For any finite CW space X the sequence $\{\mathrm{TC}_n(X)\}$ grows almost linearly with respect to n.

This follows from the inequality $\mathrm{TC}_n(X) \le \operatorname{cat} X^n \le n \operatorname{cat} X$.

Given X, we (can) introduce the power series (generating functions)

$$\mathscr{F}_X(z) = \sum_{n=1}^{\infty} \mathrm{TC}_{n+1}(X)(z^n)$$

and ask about analytical properties of them.

For example, for $X = \mathbb{S}^{2k+1}$ we have a rational function

$$\mathscr{F}_{\mathbb{S}^{2k+1}} = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}.$$

(Note my typo in [R6, Example 15.3].)

Open Question 13.2. Do the power series

$$\mathscr{F}_X(z) = \sum_{n=1}^{\infty} \mathrm{TC}_{n+1}(X) z^n$$

represent rational functions?

In [FO, Section 8] it is shown that rationality holds for many important cases. Note that for all spaces X as in [FO] we have

$$\mathscr{F}_X(z) = \frac{P_X(z)}{(1-z)^2}$$

where $P_X(z)$ is an integer polynomial with $P(1) = \operatorname{cat} X$. However, there are examples with

$$\mathscr{F}_X(z) = \frac{P_X(z)}{(1-z)^2}$$

but $P_X(1) \ne \operatorname{cat} X$, see [FKS].

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